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REMARKS ON A PAPER OF LEE AND LIM

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Abstract. Lee and Lim (2009) state three characterizations of Loamax, exponential and power function distributions, the proofs of which, are based on the solutions of certain second order non-linear differential equations. For these characterizations, they make the following statement : "Therefore there exists a unique solution of the differential equation that satisfies the given initial conditions". Although the general solution of their first differential equation is easily obtainable, they do not obtain the general solutions of the other two differential equations to ensure their claim via initial conditions. In this very short report, we present the general solutions of these equations and show that the particular solutions satisfying the initial conditions are uniquely determined to be Lomax, exponential and power function distributions respectively.

1. Introduction

In proving their Theorems 2.1-2.3, Lee and Lim (2009) come across the following non-linear second order differential equations ((3.2) , (3.7) & (3.10), pages 151-152). Here are the equations: The differential equation (3.2), which we call it (1.1)

$$\frac{d^2}{dx^2}y(x)+\frac{3\left(\frac{d}{dx}y(x)\right)^2}{1-y(x)}=0,\quad x\geq 0, \tag{1.1}$$

the differential equation (3.7) , which we call it (1.2)

$$\frac{d^2}{dx^2}y(x)+\frac{3\left(\frac{d}{dx}y(x)\right)^2}{1-y(x)}-\frac{2\left(\frac{d}{dx}y(x)\right)^3}{\lambda\left(1-y(x)\right)^2}=0,\quad x\geq 0, \tag{1.2}$$

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and the differential equation (3.10) (corrected version), which we call it (1.3)

$$\frac{d^2}{dx^2}y(x) - \frac{3\left(\frac{d}{dx}y(x)\right)^2}{y(x)} + \frac{(1-2\alpha)\ x\left(\frac{d}{dx}y(x)\right)^3}{\alpha^2\left(y(x)\right)^2} = 0, \quad x \geq 1. \tag{1.3}$$

For these differential equations, Lee and Lim make the following statement: "Therefore there exists a unique solution of the differential equation that satisfies the given initial conditions. By the existence and uniqueness theorem, we get Lomax (exponential and power function respectively) distribution." Unfortunately, they do not provide the general solutions of these equations to ensure their claim via initial conditions. Furthermore, the use of "By the existence and uniqueness theorem" is not clear at all. In what follows we provide the general solutions of the above mentioned differential equations from which Lomax, exponential and power function distributions will be uniquely determined via the given initial conditions.

2. General solutions of (1.1), (1.2) & (1.3)

The general solution of (1.1) can easily seen to be

$$y(x) = 1 - \{2(C_1 x + C_2)\}^{-1/2}. \tag{2.1}$$

Using initial conditions $y(0) = 0$ and $y'(0) = 1$, we have $y(x) = 1 - (2x+1)^{-1/2}$.

For (1.2) and (1.3) , however, finding general solutions is not as easy as for (1.1) . With the help of Maple , the general solution for (1.2) is shown to be

$$y(x) = e^{\frac{1}{2}LambertW\left(-(1+C_1\lambda)\left(e^{\lambda x}\right)^2\left(e^{C_2\lambda}\right)^2\right)-\lambda x - C_2\lambda} + 1, \tag{2.2}$$

which is in terms of Lambert W function (for further details on this function, we refer the interested reader to Brian Hayes, 2005). Using the initial conditions ($y(0) = 0$, $y'(0) = \lambda$ and $y''(0) = -\lambda^2$) and the properties of the Lambert W function , we have from (2.2) , $y(x) = e^{i\pi - \lambda x} + 1 = 1 - e^{-\lambda x}$, $x \geq 0$, since $e^{i\pi} = -1$. This particular solution with the given initial conditions is clearly unique.

Again, with the help of Maple, the general solution for (1.3) is shown to be given implicitly by

$$2\left(\frac{1-\alpha}{\alpha}\right) x\ y^{\frac{2\alpha-1}{\alpha}} - C_1\ y^{\frac{2(\alpha-1)}{\alpha}} + C_2 = 0. \tag{2.3}$$

Letting $x = 1$, $y(1) = 1$, taking implicit derivative of (2.3) and using the initial conditions ($y(1) = 1$, $y'(1) = -\alpha$) , we obtain $2\left(\frac{1-\alpha}{\alpha}\right) - C_1 + C_2 = 0$ and $C_1 = 2\left(\frac{1-\alpha}{\alpha}\right)$ respectively, which implies that $C_2 = 0$. Now, substituting for C_1 and C_2 in (2.3) , we arrive at $y(x) = x^{-\alpha}$, $x \geq 1$. This particular solution with the given initial conditions is clearly unique. It seems that the third initial condition $y''(1) = \alpha(\alpha+1)$ is not needed.

References

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